Subshifts of Finite Symbolic Rank

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- Every rank-one transformation is uniquely ergodic.

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Fact

- Every rank-one transformation is uniquely ergodic.
- The class of all rank-one transformations is a dense G_δ in the Polish space of all measure-preserving transformations.

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Problem (Weiss): Characterize all (minimal) Cantor systems which are conjugate to a rank-one subshift.

Question: Is the class of all rank-one subshifts a dense G_δ in the Polish space of all Cantor systems?

Rank-one transformations: cutting and stacking

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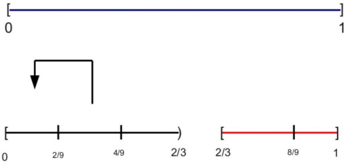
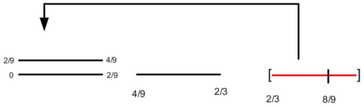


Fig. 2.1.1: Constructing Chacon's transformation at Stage 1



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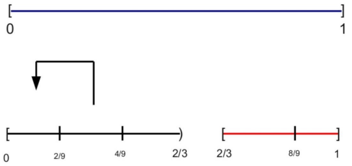

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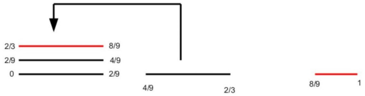
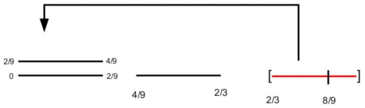


Fig. 2.1.2: Intermediate step of Chacon's process at Stage 1



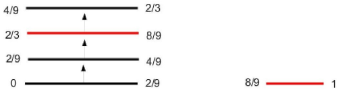


Fig. 2.1.3: End of Stage 1

Figure: Carole Agyeman-Prempeh: Chacon's transformation

Rank-one transformations: cutting and stacking

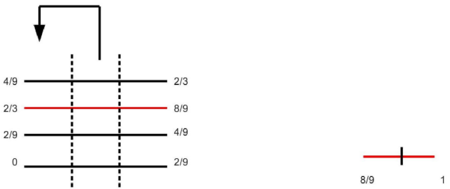


Fig. 2.1.4: Subdividing tower-1 and stacking middle portion at Stage 2

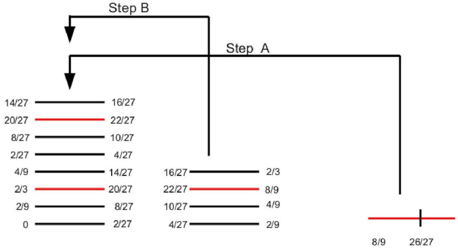


Fig. 2.1.5: Intermediate steps in Stage 2

Rank-one transformations: cutting and stacking

Fig. 2.1.6: End of Stage 2 of the Chacon process

Rank-one transformations: cutting and stacking

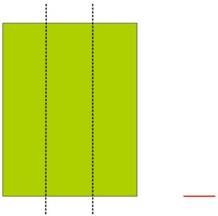


Fig. 2.1.7: Slicing and stacking the nth tower

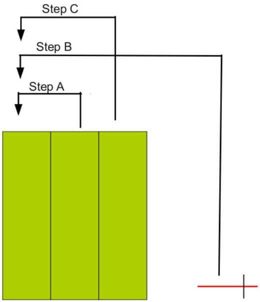


Fig. 2.1.8: Intermediate steps of $(n+1)$th stage

Rank-one transformations: cutting and stacking

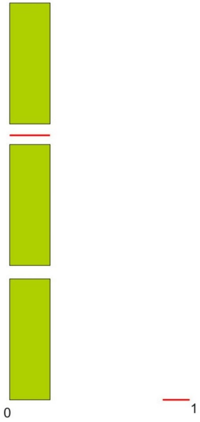


Fig. 2.1.9: End of the $(n+1)$th stage

Rank-one subshifts

Chacon's map

$$v_{n+1} = v_n v_n 1 v_n$$

$$\begin{aligned} v_0 &= 0 \\ v_1 &= 0010 \\ v_2 &= 0010\ 0010\ 1\ 0010 \\ v_3 &= 0010001010010\ 0010001010010\ 1\ 0010001010010 \\ & \ldots\ldots \end{aligned}$$

Rank-one subshifts

Given

- a sequence of positive integers $r_n > 1$ for $n \in \mathbb{N}$ (cutting parameter), and
- a doubly indexed sequence of nonnegative integers $s_{n,i}$ for $n \in \mathbb{N}$ and $0 < i < r_n$ (spacer parameter),

define a generating sequence of finite $0, 1$-words recursively by

$$\begin{aligned} v_0 &= 0 \\ v_{n+1} &= v_n 1^{s_{n,1}} v_n 1^{s_{n,2}} \cdots v_n 1^{s_{n,r_n-1}} v_n \end{aligned}$$

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An infinite rank-one word $V \in 2^{\mathbb{N}}$ is defined as $V = \lim_{n\to\infty} v_n$ and the rank-one subshift (X_V, σ) is given by

$$X_V = \{x \in 2^{\mathbb{Z}} : \text{every finite subword of } x \text{ is a subword of } V\}$$

and $\sigma(x)(k) = x(k+1)$ for all $x \in X_V$ and $k \in \mathbb{Z}$.

Rank-one subshifts

Fact: TFAE:

(1) The rank-one subshift (X_V, S) is finite (degenerate).

(2) The infinite rank-one word V is periodic.

(3) The spacer parameter is eventually constant, i.e. there is N such that for all $n, m > N$ and $0 < i < r_n$, $0 < j < r_m$, we have $s_{n,i} = s_{m,j}$.

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Fact: TFAE for a nondegenerate rank-one subshift (X_V, S):

(a) (X_V, S) is minimal.

(b) The spacer parameter is bounded, i.e., there is M such that for all $n \in \mathbb{N}$ and $0 < i < r_n$, we have $s_{n,i} \leqslant M$.

Rank-one subshifts

Theorem (G.–Ziegler, 2019)
The maximal equicontinuous factor of a rank-one subshift is finite. In particular, if (X_V, S) is minimal, then its maximal equicontinuous factor is $\mathbb{Z}/p_{\max}\mathbb{Z}$, where $p_{\max}$ is the largest p for which there is $n \in \mathbb{N}$ such that for all $m \geqslant n$ and $0 < i < r_m$, we have $p \mid (|v_n| + s_{m,i})$.

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Theorem (G.–Ziegler, 2020)
A subshift factor of a rank-one subshift (X_V, S) is either finite or isomorphic to (X_V, S).

Symbolic rank $n \geqslant 1$

(G.–Jacoby–Johnson–Leng–Li–Silva–Wu, 2023+)

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Let $\mathcal{F}$ denote the set of all finite $0, 1$-words that start and end with 0.

- For $S \subseteq \mathcal{F}$ and $w \in \mathcal{F}$, we say that w is built from S if there are $v_1, \ldots, v_{k+1} \in S$ and $s_1, \ldots, s_k \in \mathbb{N}$ such that
$$w = v_1 1^{s_1} v_2 1^{s_2} \cdots v_k 1^{s_k} v_{k+1}.$$

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$$w = v_1 1^{s_1} v_2 1^{s_2} \cdots v_k 1^{s_k} v_{k+1}.$$
- A rank-n generating sequence $v_{i,j}$ for $i \in \mathbb{N}$ and $1 \leqslant j \leqslant n_i$, where $1 \leqslant n_i \leqslant n$, satisfies
  - $v_{0,j} = 0$ for all $1 \leqslant j \leqslant n_0$
  - $v_{i+1,1}$ is built from $S_i = \{v_{i,1}, \ldots, v_{i,n_i}\}$ starting with $v_{i,1}$
  - for $2 \leqslant j \leqslant n_i$, $v_{i+1,j}$ is built from S_i

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- An infinite rank-n word $V \in 2^{\mathbb{N}}$ is defined as $V = \lim_{i \to \infty} v_{i,1}$ and a rank-$\leqslant n$ subshift (X_V, S) is defined similarly as in the rank-one case.

Symbolic rank $n \geqslant 1$

A proper rank-n generating sequence $v_{i,j}$ for $i \in \mathbb{N}$ and $1 \leqslant j \leqslant n$, satisfies

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Fact: For any $n \geqslant 1$, there is an infinite word V with a proper rank-$(n+1)$ generating sequence and no rank-n generating sequence.

Subshifts of symbolic rank $n \geqslant 1$

Theorem: TFAE for a rank-n subshift (X_V, S):

(a) (X_V, S) is minimal.

(b) V has a proper rank-n generating sequence in which the spacer parameter is bounded.

Question: What is the relationship between symbolic rank and topological rank?

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- (HPS, 1992) If (X, T) is an essentially minimal Cantor system and x_0 is in the unique minimal set, then there is an essentially simple ordered Bratteli diagram $B = (V, E, \preceq)$ with $x_0 = x_{\mathrm{min}}$ so that (X, T) is conjugate to the Vershik system (X_B, λ_B).

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- (Downarowicz–Maass, 2008; Durand, 2010) An essentially minimal Cantor system (X, T) has topological rank K if K is the minimal number such that there exists an essentially simple ordered Bratteli diagram $B = (V, E, \preceq)$ such that (X, T) is conjugate to (X_B, λ_B) and for all $i \geqslant 1$, $|V_i| \leqslant K$.

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Theorem (G.–Li, 2023+)
The following classes are G_δ subsets of the Polish space of all Cantor systems:

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5. The class of all infinite odometers.

Rank-1 subshifts revisited

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- The class of all infinite odometers is G_δ.
- The class of all minimal rank-1 subshifts is also dense.
- An infinite odometer is not conjugate to any subshift.

We answer Weiss's question by giving a chacterization of all minimal Cantor systems conjugate to a rank-1 subshift. The descriptive complexity of the characterization is apparently $\mathbf{\Sigma}^0_5$.

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Theorem (G.–Li, 2023+)
There exists a (non-minimal) rank-1 subshift whose topological rank is not finite.

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Theorem (G.–Li, 2023+)
Every minimal Cantor system of finite topological rank is either an odometer or conjugate to a minimal subshift of finite symbolic rank. Moreover, when the system has topological rank $K > 1$ and is not an odometer, it is conjugate to a subshift of symbolic rank $\leqslant K$.

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Compare

Theorem (Donoso–Durand–Maass–Petite, 2021)
Every minimal Cantor system of finite topological rank is either an odometer or conjugate to a minimal $\mathcal{S}$-adic subshift of finite alphabet rank. Moreover, when the system has topological rank $K > 1$ and is not an odometer, it is conjugate to an $\mathcal{S}$-adic subshift of alphabet rank $\leqslant K$.

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Corollary
A Cantor factor of a minimal subshift of finite symbolic rank is either an odometer or conjugate to a minimal subshift of finite symobolic rank.

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Example If V is the Chacon word, and W is obtained from V by the substitution $0 \mapsto 1$ and $1 \mapsto 0$, then X_W is a minimal subshift of finite symbolic rank (it has symbolic rank 2).

Other factors of subshifts of finite symbolic rank

Theorem (G.–Li, 2023+)

1. For any infinite odometer $(\mathcal{O}, +\mathbb{1})$, there is a minimal subshift of symbolic rank 2 whose maximal equicontinuous factor is $(\mathcal{O}, +\mathbb{1})$.

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1. For any infinite odometer $(\mathcal{O}, +\mathbb{1})$, there is a minimal subshift of symbolic rank 2 whose maximal equicontinuous factor is $(\mathcal{O}, +\mathbb{1})$.
2. For any irrational rotation $(\mathbb{T}, +\alpha)$, there is a minimal subshift of symbolic rank 2 whose maximal equicontinuous factor is $(\mathbb{T}, +\alpha)$.

Thank You!